~

Reccurence Relation

Recursion → A tech of defining a function, set, seq, Algo in the term of itself.

Recursive formula :- A formula which define any term of a sequence in terms of any number of its previous terms or which express any terms of a sequence as a function of its previous terms is called recursive and the relation is called recursive relation.

→ ~~Also called~~ the Difference Equation.

$$a_n = a_{n-1} + 1$$

Initial $a_0 = 0$

$n = 1$

$I_1 : \ a_1 = a_0 + 1$

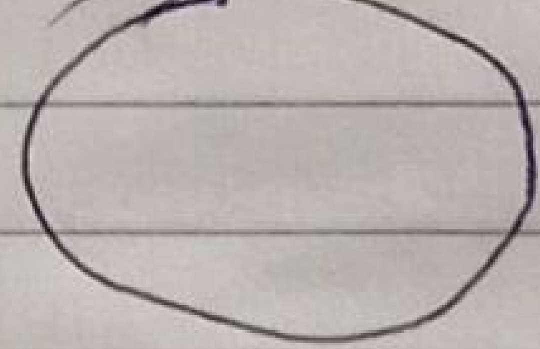
$I_2 : \ a_2 = a_1 + 1$

$\vdots \quad \vdots \quad \vdots$

$I_n = a_n = a_{n-1} + 1$ → Recurrence relation.

Ex :-

"Every hour the no. of bacteria is getting doubled".



Bacteria Colony

which one is the recurrence relation which defines the statement.

(a) $a_n = a_{n-1} + 2$ (b) $a_n = 2a_{n-1}$ ✓ (C) $a_n = a_{n-1} + a_{n-2}$

(d) None.

Recursive Formula: A

There are three steps :-

(1) Basic step : we find out the primitive term / initial Term

(2) Recursive step: We generate a formula / Rate to find the new term using exciting term.

(3) Terminal / final ÷ Verification of formula.

Ex. 2, 9, 16, 23, 30 ....

$a_1 = 2$, $a_2 = 9$, $a_3 = 16$, $a_4 = 23$ ...

$a_2 - a_1 = 7$

$a_3 - a_2 = 7$ ⇒ Generalize

$a_4 - a_3 = 7$

$a_n - a_{n-1} = 7$ $a_1 = 2, n \geq$

$\boxed{a_n = a_{n-1} + 7}$

Recurrence Formula

(step by step)

$a_n = a_{n-1} + 7$; $n \geq 2$, $a_1 = 2$

$n = 2$

$a_2 = a_{2-1} + 7$

$= a_1 + 7$

$= 2 + 7 = 9$

$n = 3$

$a_3 = a_2 + 7$

$= 9 + 7$

$= 16$

General Formula

(Direct)

$a = 2$ $d = 7$

$t_n = a + (n-1)d$

$a_n = 2 + (n-1)7$

$\boxed{a_n = 7n - 5}$ $n \in I$

Ex:- find a_{10}. $n = 10$

$a_{10} = 7(10) - 5$

$= 65$

find $a_{10} = a_9 + 7$

To find a_{10}, we have to find a_9. So, we canot find any term directly so you should move step by step.

In general formula, we directly move to the term we want to find.

Fibonacci Series :- 0, 1, 1, 2, 3 . . .

Primitive/initial

$a_1 = 0$ or $a_0 = 0$

$a_2 = 1$ $a_1 = 1$

$a_n = a_{n-1} + a_{n-2}$: $n \geq 3$

$a_{n+1} = a_n + a_{n-1}$: $n \geq 2$

Note: The recurrene relation can be written in different form of equation

Ex:-

$a_n = a_{n-1} + 5$ $a_n + a_{n-1} - 2a_{n-2} = 0$

$t_n = t_{n-1} + 5$ $y_n + y_{n-1} + 2y_{n-2} = 5$

$f_n = f_{n-1} = 5$ $f_n + f_{n-1} + 2f_{n-2} = 5$

$f_n = f_{n-1} + 5$ $S_n + S_{n-1} + 2S_{n-2} = 5$

Order of recurrence relation - The order of recurrence relation or difference equation is defined to be the difference between highest and lowest subscripts of the $f(n)$, or a_n or y_n or $f(x)$ etc.

Ex $y_n + y_{n-1} - 3y_{n-2} + 6y_{n-3} = 5$

Highest $= n$

Lowest $= n-3$

Order $= H - L$

$= n - (n-3)$

$= 3$

Scanned by TapScanner

So, this is a Third order reccurence relation.

Degree of recurrence relation or Difference equation :-

The degree of the recurrence relation is defined to be the highest power of $f(n)$ or a_n or s_n or $f(x)$ or y_n.

Ex:- $y^3_{r+3} - 5y_{r+2} + 3y_r - 2y_{r-1} = 0$

Order $= (r+3) - (r-1) = 4.$

So, this is a 4^{th} Order recurrence relation.

Degree - 3.

Note: If the degree of recurrence relation is

1) One, it is called linear RR.
2) Two, it is called quad RR.
3) Three, " " " Cubic RR
4) Four, " " " Biquadratic RR.

Homogeneous and Non-homogeneous RR. :- A recurrence relation is called Homogeneous RR if it contain no term that depends upon only on variable 'n' or r Otherwise it is called Non-homogeneous RR.

Linear reccurence relation with constant coefficient

A recurrence relation of the form $a_1 y_n + a_2 y_{n-1} + \dots = f(n)$ -① where $a_1 \dots a_n$ are cons. called coefficient of R.R $f(n)$ is a function of n only. is called a linear RR with cons. coeff.

If $f(n)=0$, It is called homogeneous rr.
$f(n) \neq 0$, It is called non- homo. RR.

Ex $a_n = 2a_{n-1} + 5a_{n-2} + 6a_{n-3} + 9$

a) Homogeneous ✓ b) Non-homogeneous.

Bcoz $f(n) \neq 0$. { but linear }

Ex. $a_n - \cancel{5a_n}\, 5n a_{n-2} + 6n^2 a_{n-3} = 0$

✓ a) Homogeneous b) Non-homogeneous.

Bcoz $f(n) = 0$. { but not linear }

Ex. $a_n - a_{n-2} + 6a_{n-3} = n^2$

→ Cons. coff and $f(n) \neq 0$
∴ ~~Non~~ linear non-homogeneous RR

Method to find the solution of linear, homogeneous R.R.

① Iteration Method :- In this method the recurrence relation ~~of RR~~ for a_n is used repeatedly to solve for a general solution for a_n in terms of 'n'.

Q Solve the reccurrence Relation $a_n = a_{n-1} + 3 \ \forall n \geq 1$ where $a_0 = 2$ by iteration method.

Solution :- Here,

$a_0 = 2$ { Primitive }

$a_0 \ a_1 \ a_2 \ a_3 \ \ldots \ a_{n-1} \ a_{n-1} \ a_n$

$a_n - a_{n-1} = 3 \quad ; n \geq 1$

$a_n = 3 + a_{n-1}$.

{ we want a relationship b/w a_n and a_0.

$a_{n-1} = a_{n-2} + 3$

$a_{n-2} = a_{n-3} + 3$

$a_{n-3} = a_{n-4} + 3$

$a_n = a_{n-1} + 3$

$a_n = [a_{n-2} + 3] + 3$

$a_n = [a_{n-2}] + 2 \times 3$

$a_n = [a_{n-3} + 3] + 2 \times 3$

$= [a_{n-3}] + 3 \times 3$

$a_n = [a_{n-4}] + 4 \times 3$

$\vdots \qquad \vdots$ After iteration

$a_n = [a_{n-i}] + i \times 3$

(A) the term $a_{n-i} = 0 \Rightarrow i = n$

$a_n = a_{n-n} + 3 \times n$

$a_n = a_0 + n \times 3$

$a_n = 2 + 3n$ ($a_0 = 2$ Given)

required solⁿ of given RR.

8.4

$a_n = a_{n-1} + 3 \quad n \geq 1$

$a_0 = 2$

$n=1 \quad a_1 = a_0 + 3 = 2+3 = 5$

$n=2 \quad a_2 = a_1 + 3 = 5+3 = 8$

$n=3 \quad a_3 = a_2 + 3 = 8+3 = 11$

$2, 5, 8, 11, 14 \ldots$

$a_n = 3n + 3$

$a_0 = 2$

$a_1 = \cancel{5\times 1}\ 3\times 1 + 2 = 5$

$a_2 = 3\times 2 + 2 = 8$

A.P

$a_n - a_{n-1} = \underline{k}$

K = comm. diff. of A.P

$a_n = k + a_{n-1}$

$$a_n = \begin{cases} a + (n-1)d \; ; & n \geq 1 \quad \text{i.e., } 1^{st} \text{ term} = a_1 \\ a + nd \; ; & n \geq 0 \quad \text{ie, } 1^{st} \text{ term} = a_0. \end{cases}$$

$a_n = a_{n-1} + 5$ here $k = 5$ $a_0 = 3$ $a_n = a + nd$

$= 3 + 5n.$

G.P.

$\frac{a_n}{a_{n-1}} = K \implies a_n = K \cdot a_{n-1}$

$$a_n = \begin{cases} ar^{n-1} \; ; & n \geq 1 \quad 1^{st} \text{ term} = a_1 \\ ar^n \; ; & n \geq 0 \quad 1^{st} \text{ term} = a_0 \end{cases}$$

Ex $a_n = 3a_{n-1} + 5; \quad n \geq 2 \quad a_1 = 2$

$a_n = 3\underbrace{a_{n-1}} + 5$ —— (I) $\qquad a_n = 3a_{n-1} + 5$

$\qquad a_{n-1} = 3a_{n-2} + 5$

$a_n = 3[3a_{n-2} + 5] + 5 \qquad a_{n-2} = 3a_{n-3} + 5$

$a_n = 3^2 a_{n-2} + (1+3) \times 5$ —— (II)

$a_n = 3^2[3a_{n-3} + 5] + 5(1+3) = 3^3 a_{n-3} + 5(1+3+3^2)$ —— (III)

$a_n = 3^4 a_{n-4} + 5(1+3+3^2+3^3)$

⋮

$a_n = 3^i a_{n-i} + 5(1+3+3^2 \ldots \; i \text{ terms})$

then a_{n-i} will be a_1 if $n-i = 1$

$i = n-1$

$a_n = 3^{n-1} a_1 + 5[1+3+\ldots (n-1) \text{ terms}]$

$= 3^{n-1} \times 2 + 5$ [Sum of $(n-1)$ terms of G.P $a=1, r=3$]

$S_n = \dfrac{a(r^n - 1)}{r-1}$

$= 2 \times 3^{n-1} + 5\left[\dfrac{3^{n-1} - 1}{3-1}\right]$

$S_{n-1} = \dfrac{a(r^{n-1} - 1)}{r-1}$

$= 2 \times 3^{n-1} + \dfrac{5}{2} 3^{n-1} - \dfrac{5}{2}$

$a_n = \dfrac{9}{2} \times 3^{n-1} - \dfrac{5}{2}, \quad n \geq 1$

Some useful Result

(1) Sum of n terms of an A.P with first term 'a' and common difference 'd'.

$$S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+l] \quad l - \text{last term}$$

(2) Sum of n terms of a GP

$a, ar, ar^2 \dots$

$$S_n = \frac{a(r^n-1)}{r-1}$$

(3) $\Sigma n = 1+2+3+\dots n = \dfrac{n(n+1)}{2}$

(4) $\Sigma n^2 = 1^2+2^2+3^2+\dots n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Linear homogeneous recurrence relation

A RR of the form

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} = f(n)$$

where $C_0, C_1, \dots C_k$ are constants called coefficient and $f(n)$ is a function 'n' only. is called linear recurrence relation with cons. coefficient

If $f(n) = 0$, it is called Homo RR
$f(n) \neq 0$ Non-homo RR

Method-2. Method of characteristic equation

Working rule: $a_n = C_0 a_{n-1} + C_1 a_{n-2} + \dots C_k a_{n-k} + \dots$

Step I Transpose all the terms to LHS and write the RR in the form and be sure the equation is complete.

$$a_n - C_0 a_{n-1} - C_1 a_{n-2} + \dots C_k a_{n-k} = 0$$

Step II: Find the order of RR

here the order is $n-(n-k) = k$

Step III: Write down the characteristic equation

$$x^k - C_0 x^{k-1} - C_1 x^{n-2} - C_3 x^{n-3} \dots C_k x^{n-k} = 0 \quad (1)$$

Replace the term

$[a_n$ by $x^n]$ $[a_{n-1}$ by $x^{k-1}]$ $[a_{n-2}$ by $x^{k-2}]$ ...

Step IV Solve the characteristic equation and find the roots of CE i.e, the value of of x

Step V (a): If all the roots of C.E are real and distinct say, $x = \alpha_1, \alpha_2, \alpha_3 \dots$ then linear independent solution and then the general solution

$$a_n = C_1(\alpha_1)^n + C_2(\alpha_2)^{n-1} + C_3(\alpha_3)^{n-2} + \dots$$

where $C_1, C_2, C_3 \dots$ are arbitrary constant.

By ~~giving~~ using the given internal condition we can find the values of $C_1, C_2 \dots C_k$.

(b) If two roots of C.E are equal say α, α then the corresponding general solution is $(C_1 + C_2 n)\alpha^n$.

Note: Corresponding L.I solution will be

$\alpha^n, n\alpha^n$ then General Sol^n

$$a_n = C_1\alpha^n + C_2 n\alpha^n$$
$$= \alpha^n (C_1 + nC_2)$$

(c) In case of complex roots i.e; the roots of the form $\alpha + i\beta$ and $\alpha - i\beta$ or $\alpha \pm i\beta$.

the corresponding General solution will be

$$\boxed{a_n = \rho^n [C_1 \cos\theta n + C_2 \sin\theta n]}$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$ $\tan\theta = \frac{\beta}{\alpha}$ $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$.

$\alpha = \rho\cos\theta$ $\beta = \rho\sin\theta$

Ex Solve the 2nd order recurrence relation

$$a_n = 10a_{n-1} - 9a_{n-2}; \quad n \geq 2 \qquad a_0 = 3, a_1 = 11$$

Order $= n - (n-2) = 2$

Char eqn. $x^2 - 10x + 9 = 0$

$(x-9)(x-1) = 0 \Rightarrow x = 1, 9$

$x = 1, 9$ roots are real and distinct.

$a_n = c_1(1)^n + c_2(9)^n = c_1 + c_2 9^n$

$a_n = c_1 + c_2 9^n$ $\qquad a_0 = 3, a_1 = 11$

$a_0 = c_1 + c_2 \times 9^0 = 3$
$\Rightarrow c_1 + c_2 = 3$ — (I)

$a_1 = c_1 + c_2 \times 9$
$11 = c_1 + 9c_2$ — (II)

Solving (I) and (II)

$c_1 + 9c_2 = 11$
$-c_1 - c_2 = -3$
$8c_2 = 8$
$c_2 = 1$ $\qquad c_1 = 11 - 9 = 2$

$\therefore$ Soln:- $\boxed{a_n = 2 + 9^n}$ $\quad n \geq 0$

Generating Function : If $S = \{a_0, a_1, \ldots\}$ is a sequence of real or complex numbers, then the power series given by

$$G(S, z) = G(z) = \sum_{i=0}^{\infty} a_i z^i = a_0 + a_1 z + a_2 z^2 + \ldots$$

is called the generating function for the given sequence.

$G(S, z) = G(z) = \sum_{n=0}^{\infty} a_n z^n$ where a_n is the general term of the sequence.

$$G(S, z) = G(z) = \sum_{n=0}^{\infty} a_n z^n$$

Scanned by TapScanner

Example: For the sequence of $S = \{1^2, 2^2, 3^2 \ldots\}$ the generating function is

$$G(S, Z) = G(Z) = 1^2 \cdot z^0 + 2^2 z^1 + 3^2 z^2 + \ldots$$

$$= \sum_{n=0}^{\infty} (n+1)^2 z^n \quad \text{or} \quad \sum_{n=1}^{\infty} n^2 z^{n-1}$$

Note! We can write / take the parameter 'x' at the place of parameter 'z', then the ~~geometri~~ generating function $G(x)$.

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

Generating Function of Some Standard Function

(1) $a_n = a$; $n \geq 0$ $\quad G(z) = a\left(\dfrac{1}{1-z}\right) = G(a, z)$

(2) $a_n = 1$ $\quad n \geq 0$ $\quad G(z) = \dfrac{1}{1-z} = G(1, z)$

(3) $a_n = a^n$ $\quad n \geq 0$ $\quad G(z) = \dfrac{1}{1-az} = G(a^n, z)$

(4) $a_n = n$ $\quad n \geq 0$ $\quad G(z) = \dfrac{z}{(1-z)^2} = G(n, z)$

(5) $a_n = n^2$ $\quad n \geq 0$ $\quad G(z) = \dfrac{z(z+1)}{(1-z)^2} = G(n^2, z)$

(6) $a_n = n(n+1)$; $\quad G(z) = \dfrac{2z}{(1-z)^3} = G(n(n+1), z)$

(7) $a_n = {}^mC_n$; $\quad G(z) = (1+z)^m = G({}^mC_r, z)$

(8) $a_n = (n+1)(n+2)$; $\quad G(z) = \dfrac{2}{(1-z)^3} = G(a_n, z)$

(1) $a_n = a$

$= a, a, a, a \dots$

$G(z) = az^0 + az^1 + az^2 + \dots$

$= a + az + az^2 + \dots$

$= a(1 + z + z^2 + \dots)$

$= a(1-z)^{-1} = \dfrac{a}{1-z}$

(2) $a_n = 1$

$\Rightarrow G(x) = \dfrac{1}{1-z}$

(3) $a_n = a^n$

$G(z) = a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots$

$= (az)^0 + (az)^1 + (az)^2 + \dots$ let $az = t$

$= \dfrac{1}{1-az}$

Methods to find the Generating function from RR

Let the recurrence relation be

$$S_n + C_1 S_{n-1} + C_2 S_{n-2} + \dots + C_n S_{n-r} = 0 \quad \text{--- (1)}$$

Step I Multiply both side by z^n and sum up the terms from $n = r$ to $n = \infty$.

$$\sum S_n z^n + C_1 \sum S_{n-1} z^n + C_2 \sum S_{n-2} z^n + \dots + C_r \sum S_{n-r} z^r = 0$$

Step II If $G(z) = \sum_{n=0}^{\infty} S_n z^n$ be the generating function then write each term in terms of $G(S, z)$.

Step III. Solve the equation for $G(z)$.

Some important operations

① If s and t are two sequence of natural number n then

i) $(s+t)(n) = s(n) + t(n)$

ii) $k s(n) = (ks)(n)$

iii) $(st)(n) = s(n) \cdot t(n)$

2 Convolutilation operator

$$(s * t)(n) = \sum_{r=0}^{\infty} s(r) \cdot t(n-r)$$

③ Pop operation $s\uparrow$ (s. pop)

$$s(\uparrow)n = s(n+1)$$

④ push operation $s\downarrow$ (s push)

$$s(\downarrow)(n) = \begin{cases} s(n-1) \; ; \; n>0 \\ 0 \; ; \; n=0 \end{cases}$$

⑤ $s\uparrow n = [s\uparrow(n-1)]\uparrow$ and $s\uparrow 1 = s\uparrow$

⑥ $s\downarrow n = [s\downarrow(n-1)]\downarrow$ and $s\downarrow 1 = s\downarrow$

Some important results

(1) $G(s+t, z) = G(s,z) + G(t,z)$

(2) $G(ks, z) = kG(s,z)$ $\quad k$ - constant

(3) $G(s*t, z) = G(s,z)\, G(t,z)$

(4) $G(s\uparrow, z) = \dfrac{G(s,z) - s(0)}{z}$

(5) $G(s\downarrow, z) = zG(s,z)$

(6) $G(s\uparrow n, z) = \dfrac{G(s,z) - \sum_{k=0}^{n-1} s(k) z^k}{z^n}$

(7) $G(s\downarrow n, z) = z^n G(s,z)$

Eg $\quad s\downarrow n = s[\downarrow(n-1)]\downarrow$ and $s\downarrow I = s\downarrow$

$$\begin{aligned}
(s\uparrow 3)(n) &= (s\uparrow(2)\uparrow)(n) \\
&= ([s(\uparrow)\uparrow]\uparrow)(n) \\
&= (((s\uparrow)\uparrow)\uparrow)(n) \\
&= ((s\uparrow)\uparrow)(n+1) \\
&= s\uparrow(n+2) \\
&= s(n+3)
\end{aligned}$$

$\therefore (s\uparrow 3)(n) = s(n+3)$

$(s\uparrow k)(n) = s(n+k)$

Also $\begin{cases} (s\downarrow k)(n) = s(n-k); & n \geq k \\ \quad = 0 & ; n < k \end{cases}$

Q. Find the generating function of the sequence

$S_n = 3\cdot4^n + 2(-1)^n + 7$

Solⁿ

$S_n = 3\cdot4^n + 2(-1)^n + 7$ (Given)

$= 3\cdot4^n + 2(-1)^n + 7\cdot1^n$

$\Rightarrow G(S,z) = 3G(4)^n + 2G(-1)^n + 7G(1)^n$ $\left\{ G(kS,z) = kG(S,z) \right.$

$= 3\left(\frac{1}{1-4z}\right) + 2\left(\frac{1}{1-(-1)z}\right) + 7\left(\frac{1}{1-z}\right)$

$= \frac{3}{1-4z} + \frac{2}{1+z} + \frac{7}{1-z}$ $\left\{ \begin{array}{l} a_n = a^n \\ Z(a,z) = \frac{1}{1-az} \end{array} \right.$

$= \frac{3(1-z^2) + 2(1-4z)(1-z) + 7(1+z)(1-4z)}{(1-4z)(1+z)(1-z)}$

$= \frac{3(1-z^2) + 2(1-5z+4z^2) + 7(1-3z-4z^2)}{(1-4z)(1-z^2)}$

$= \frac{3-3z^2+2-10z+8z^2+7-21z-28z^2}{(1-4z)(1-z^2)}$

$= \frac{12-31z-23z^2}{(1-4z)(1-z^2)}$

Ex Find the generating function of the sequence.

$\therefore$ ~~ex~~ $s_n = 7\cdot6^n - 5^n$

Sol^n $s_n = 7\cdot6^n - 5^n$ [G(~~s~~ s, ~~z~~)

$$G(s_n, z) = 7\cdot\left(\frac{1}{1-6z}\right) - \left(\frac{1}{1-5z}\right)$$

$$= \frac{7(1-5z) - (1-6z)}{(1-6z)(1-5z)}$$

$$= \frac{6 - 29x}{1-11x+30z^2}$$

Ex For the recurrence relation

$a_n = 6a_{n-1} - 8a_{n-2}$; $n \geq 2$ $a_0 = 10$, $a_1 = 25$

find the generating function of the RR and also the sol^n.

Sol^n:- The given recurrence relation is

$a_n = 6a_{n-1} - 8a_{n-2}$; $n \geq 2$ $a_0 = 10$, $a_1 = 25$

$\Rightarrow a_n - 6a_{n-1} + 8a_{n-2} = 0$

Multiply both side by z^n and summing up from $n = 2$ to ∞

$$\Rightarrow \sum_{n=2}^{\infty} a_n z^n - 6\sum_{n=2}^{\infty} a_{n-1} z^n + 8\sum_{n=2}^{\infty} a_{n-2} z^n = 0$$

$$= \left(\sum_{n=0}^{\infty} a_n z^n - a_1 z - a_0\right) - 6z\sum_{n=2}^{\infty} a_{n-1} z^{n-1} + 8z^2\sum_{n=2}^{\infty} a_{n-2} z^{n-2}$$

Scanned by TapScanner

$$\left(\sum_{n=0}^{\infty} a_n z^n - a_1 z - a_0\right) - 6z\left(\sum_{n=0}^{\infty} a_{n-1} z^{n-1} - a_0\right)$$

$$+ 8z^2 \sum_{n=0}^{\infty} a_n z^n = 0.$$

$$\left(G(a,z) - 10 - 25z\right) - 6z\left(G(a,z) - 10\right) + 8z^2 G(a,z).$$
$$= 0.$$

$\Rightarrow$ ~~G(a,z) [1 - 10 - 25z - 6z + 60z + 8z^2] = 0~~

$$= G(a,z)\left[1 - 6z + 8z^2\right] = 10 - 35z$$

$$G(a,z) = \frac{10-35z}{1-6z+8z^2} = \frac{10-35z}{1-4z-2z+8z^2}$$

$$G(a,z) = \frac{10-35z}{(1-4z) - 2z(1-4z)} = \frac{10-35z}{(1-4z)(1-2z)}$$

$$= \cancel{10-35z}\ \frac{5/2}{(1-4z)} + \frac{15/2}{(1-2z)}$$

$$\Rightarrow \frac{5}{2}G(4^n) + \frac{15}{2}(2)^n$$

$$= G\left[\frac{5}{2}\cdot 4^n + \frac{15}{2}\cdot 2^n\right]$$

$$a_n = \frac{5}{2}\cdot 4^n + \frac{15}{2}2^n \qquad n \geq 0$$

Note #

$$6\sum_{n=2}^{\infty} a_{n-1} z^n = 6z \sum_{n=2} a_{n-1} z^{n-1}$$

$$= 6z\left[a_1 z^1 + a_2 z^2 + \dots + a_0 z^0 - a_0\right]$$

$$= 6z\left[G(a,z) - a_0\right].$$

Operator Method:

Ex $a_n = 7a_{n-1} - 12a_{n-2} + 5^n$

Δ - Difference operator
E - shift operator

→ Non-homogeneous equation.

write homogeneous part only —

$n \geq 2$.

$a_n = 7a_{n-1} - 12a_{n-2}$

$\Rightarrow r^2 = 7r - 12$

$\Rightarrow r^2 - 7r + 12 = 0$

$\Rightarrow (r-4)(r-3) = 0$

$\Rightarrow r = 4, 3$

CF $\quad c_1(4)^n + c_2(3)^n$

$a_n = 7a_{n-1} - 12a_{n-2}$

$a_n - 7a_{n-1} + 12a_{n-2} = 0$

lowest subscript = k

Put $n-2 = k \Rightarrow n = k+2$

$a_{k+2} - 7a_{k+1} + 12a_k = 5^{k+2}$

Imp

$$\boxed{\begin{aligned} &\Delta f(x) = f(x+1) - f(x) \\ &\text{Ex: } \Delta k = (k+1) - (k) \\ &\Delta x^2 = (x^2+1)^2 - (x^2) \\ &\Delta(\text{const}) = 0 \end{aligned}}$$

k - variable

$\Delta(k^2+k) = [(k+1)^2 + (k+1)] - (k^2+$

$= 2k+1$

$$\boxed{\begin{aligned} &E f(x) = f(x+1) \\ &E(k) = k+1 \\ &E(k^2+k) = (k+1)^2 + (k+1) \end{aligned}}$$

$E(\text{const}) = \text{cons.}$

To find CF

$E^2 - 7E + 12 = 0$

$(E-4)(E-3) = 0$

$E = 4, 3.$

$CF = C_1(4)^n + C_2(3)^n$

To find PI

$\frac{1}{E^2 - 7E + 12}(5^{k+2}) = 5^2 \frac{5^k}{E^2 - 7E + 12}$

Here $a = 5$

$25 \frac{5^k}{(5)^2 - 7(5) + 12}$ $\cancel{\ }$ $= \frac{5^{k+2}}{2}$

$\frac{25}{2} \times 5^{n-2}$ $(k = n-2)$

Rule of P.I:

Case.

1) If RHS ~~eq~~ is in a^n form, then put $E = 'a'$. provided $f(E) \neq 0$.

$f(E)$ is the denominator term.

Case II If the RHS is any form of polynomial.

Ex:- Previous Example.

$a_n = 7a_{n-1} - 12a_{n-2} + (n-2)$

$a_n - 7a_{n-1} + 12a_{n-2} = (n-2)$

Put $n-2 = k$,

$a_{k+2} - 7a_{k+1} + 12k = k$

$CF - C_1(4)^n + C_2(3)^n$

$PI = \frac{1}{E^2 - 7k + 12}(k) = \frac{1}{12\left[1 + \frac{E^2 - 7k}{12}\right]}(k)$

$$\frac{1}{12}\times\left[1+\frac{E^2-7E}{12}\right]^{-1}(k)$$

$$=\frac{1}{12}[1+a]^{-1}(k) \qquad a=\frac{E^2-7E}{12}$$

$$\Rightarrow \frac{1}{12}[1-a+a^2-a^3+a^4-\dots](k)$$

Note:-
If the power of ~~follow~~ polynomial is k^n, then take terms upto the ~~n power~~ degree of given polynomial

Here,
Degree = 1
~~Power of Δ~~
So, we take upto a only

$$\frac{1}{12}(1-a)(k) = \frac{1}{12}\cancel{[k-a]}$$

$$\Delta = \frac{d}{dx}$$

$$=\frac{1}{12}\left[1-\frac{E^2-7E}{12}\right](k) \qquad E=1+\Delta$$

$$=\frac{1}{12}\left[1-\frac{(1+\Delta)^2-7(1+\Delta)}{12}\right](k)$$

$$=\frac{1}{12}\left[1-\frac{1}{12}(\Delta^2-5\Delta-6)\right]$$

$$=\frac{1}{12}\left[k-\frac{1}{12}[0-5\times1-6k]\right]$$

$$=\frac{1}{12}\left[k+\frac{5+6k}{12}\right]$$

$$=\frac{1}{12}\left[(n-2)+\frac{1}{12}[5+6(n-2)]\right]$$

Scanned by TapScanner

Ex

Case III If RHS is of the form e^{ax}. { convert e^{ax} in form of b^n, $\Rightarrow b = e^a$ use rule 1 to find PI.

Ex $a_{n+1} - 5a_n = e^{2n}$

Solⁿ CF+PI

$LI = n.$

∵ all index are +ve no need to change variable

$(E-5) = e^{2n}.$

CF $E-5=0$

$E=5$

∴ CF is $C_1(5)^n$

PI $\frac{1}{E-5} e^{2n}$

convert

$e^{2n} = a^n$

$\Rightarrow a = e^2$

$$\boxed{\frac{1}{e^2-5} \times e^{2n}.}$$

∴ $Sol^n = CF + PI = C_1(5)^n + \frac{e^{2n}}{e^2-5}.$ Ans

Case IV RHS = sin or cos function { convert into exponential then apply rule 3 }

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

cos, var index

$\cos an = \frac{e^{i(an)} + e^{-i(an)}}{2}$ $\sin\theta = \frac{e^{i(an)} - e^{-i(an)}}{2}$

Ex

Previous example

Ex $PI = \frac{1}{E-5} \sin 2x$

$= \frac{1}{E-5} \left[\frac{e^{2ix} - e^{-2ix}}{2i}\right]$

$= \frac{1}{2i}\left[\frac{1}{E-5}\left[e^{2ix} - e^{-2ix}\right]\right] = \frac{1}{2i}\left[\frac{1}{E-5}(e^{2i})^{x} - \frac{1}{E-5}\right.$

$= \frac{1}{2i}\left(\frac{1}{e^{2i}-5} e^{2ix} - \frac{1}{e^{-2i}-5} e^{-2ix}\right).$